Further analogs of Andrews' spt-function identity

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What we study

Main work

We give new analogues of Andrews' celebrated spt-function identity. Instead of using Watson's ${}_8\phi_7$ transformation or Bailey-type ${}_{10}\phi_9$ transformations, we use two ${}_{12}\phi_{11}$ transformations of Verma and Jain between basic hypergeometric series on different bases.

Form of the results

Each new identity has the same structure:

$$\text{weighted smallest parts-type series} = \text{Lambert series term} + \text{oscillating term}.$$

Takeaway

The work expands the catalogue of spt-type identities and suggests possible links with mock theta functions and other restricted partitions.

Contents

1. Review the smallest parts function spt(n) and Andrews' original identity.
2. Recall previous analogues connected with $\omega(q)$, $\nu(q)$, and $f(q)$.
3. State the two new Verma-Jain transformations used.
4. State the two new identities.
5. Explain the proof mechanism - specialization, limit, rearrangement, and second derivative at $z = 1$.
6. Discuss the concluding observations - mock theta questions, further possible analogues, and structural symmetry.

Partitions and the spt-function

A partition $\lambda \vdash n$ is a nonincreasing sequence of positive integers whose sum is n. Andrews' smallest-parts statistic introduced in 2008 counts the total multiplicity of the smallest part across all partitions.

$$\mathrm{spt}(n) = \sum_{\lambda \vdash n} \#\{\text{parts of } \lambda \text{ equal to the smallest part of } \lambda\}.$$

Example ($n = 4$)

The partitions of 4 are

$$4, \qquad 3+1, \qquad 2+2, \qquad 2+1+1, \qquad 1+1+1+1.$$

The smallest-part multiplicities are $1, 1, 2, 2, 4$. Hence $\mathrm{spt}(4) = 10$.

It has striking Ramanujan-type congruences -

$$\mathrm{spt}(5n+4) \equiv 0 \pmod 5,$$

$$\mathrm{spt}(7n+5) \equiv 0 \pmod 7, \qquad \mathrm{spt}(13n+6) \equiv 0 \pmod{13}.$$

Notation

q-series (Eulerian series, basic hypergeometric series, q-hypergeometric series) are constructed from the q-rising factorials (q-Pochhammer symbols).

The usual notation for the conventional q-Pochammer symbols are

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),\ (a)_\infty = (a;q)_\infty := \lim_{n\to\infty}(a;q)_n,$$
$$(a_1,a_2,\dots,a_j;q)_n = (a_1;q)_n(a_2;q)_n\dots(a_j;q)_n,$$
$$(a_1,a_2,\dots,a_j;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\dots(a_j;q)_\infty.$$

Generically, q-series take the form ($|q|<1$)

$$\sum_{n\geq 0}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n z^n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}.$$

$p(n)$ - number of integer partitions of n.

Generating function for spt(n)

A standard combinatorial argument gives

$$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_\infty}.$$

- Choose the smallest part to be n.
- The number of smallest parts contributes $q^n + 2q^{2n} + 3q^{3n} + \cdots = \dfrac{q^n}{(1-q^n)^2}$.
- Parts larger than n are unrestricted and contribute $(q^{n+1};q)_\infty^{-1}$.

Andrews' original spt identity (Crelle journal, 2008)

Andrews' identity is

$$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{n q^n}{1-q^n} + \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n)}{(1-q^n)^2} q^{n(3n+1)/2}.$$

It is obtained using Watson's q-analogue of Whipple's theorem, a ${}_8\phi_7$ transformation.

$$ {}_8\phi_7\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ q^{-N} \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq^{N+1} \end{matrix}; q,\ \frac{a^2 q^{N+1}}{bcde} \right)$$

$$= \frac{(aq,\ aq/de;q)_N}{(aq/d,\ aq/e;q)_N}\ {}_4\phi_3\left(\begin{matrix} aq/bc,\ d,\ e,\ q^{-N} \\ aq/b,\ aq/c,\ deq^{-N}/a \end{matrix}; q,\ q \right).$$

Rank moment interpretation in Andrews' identity

The first term on the right is the generating function for $np(n)$, where $p(n)$ is the partition function -

$$\frac{1}{(q;q)_\infty} \sum_{n \geq 1} \frac{nq^n}{1-q^n}.$$

The second term is the generating function for $\frac{1}{2}N_2(n)$, where

$$N_2(n) = \sum_{m=-\infty}^{\infty} m^2 N(m,n).$$

Here $N(m,n)$ counts partitions of n with rank m.

Rank - statistic due to Freeman Dyson (Eureka, 1944)

Rank of a partition is the largest part minus the number of parts. As in, for $5+3+3+1$, the largest part is 5, the number of parts is 4, and the rank is $5-4=1$.

Andrews' original spt identity

Andrews' identity is

$$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{n q^n}{1-q^n} + \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n)}{(1-q^n)^2} q^{3n(n+1)/2}.$$

It is obtained using Watson's q-analogue of Whipple's theorem, an ${}_8\phi_7$ transformation.

$$ {}_8\phi_7\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ q^{-N} \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq^{N+1} \end{matrix}; q,\ \frac{a^2 q^{N+1}}{bcde} \right)$$

$$= \frac{(aq,\ aq/de;q)_N}{(aq/d,\ aq/e;q)_N}\ {}_4\phi_3\left(\begin{matrix} aq/bc,\ d,\ e,\ q^{-N} \\ aq/b,\ aq/c,\ deq^{-N}/a \end{matrix}; q,\ q \right).$$

Analogue due to Andrews-Dixit-Yee (Res. Number Theory, 2015)

Andrews, Dixit, and Yee proved an analogue related to $q\omega(q)$ and $\nu(-q)$

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_n(q^{2n+2};q^2)_\infty} = \frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^{2n})}{(1-q^{2n})^2}q^{n(3n+1)}.$$

Context

This came from a limiting case of a ${}_{10}\phi_9$ transformation due to Bailey, in a form used by Andrews.

$$\lim_{N\to\infty} {}_{10}\phi_9\left(\begin{matrix} a,\ q^2\sqrt{a},\ -q^2\sqrt{a},\ p_1,\ p_1q,\ p_2,\ p_2q,\ f,\ q^{-2N},\ q^{-2N+1} \\ \sqrt{a},\ -\sqrt{a},\ \frac{aq^2}{p_1},\ \frac{aq}{p_1},\ \frac{aq^2}{p_2},\ \frac{aq}{p_2},\ \frac{aq^2}{f},\ aq^{2N+2},\ aq^{2N+1} \end{matrix}; q^2,\ \frac{a^3q^{4N+3}}{p_1^2p_2^2f}\right)$$

$$= \frac{(aq;q)_\infty\left(\frac{aq}{p_1p_2};q\right)_\infty}{\left(\frac{aq}{p_1};q\right)_\infty\left(\frac{aq}{p_2};q\right)_\infty}\sum_{n=0}^{\infty}\frac{(p_1;q)_n(p_2;q)_n\left(\frac{aq}{f};q^2\right)_n}{(q;q)_n(aq;q^2)_n\left(\frac{aq}{f};q\right)_n}\left(\frac{aq}{p_1p_2}\right)^n.$$

The statistic $\mathrm{spt}_\omega(n)$

This analogue has a partition-theoretic statistic behind it.

$$\sum_{n=1}^{\infty} \mathrm{spt}_\omega(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_n (q^{2n+2};q^2)_\infty}$$

The function $\mathrm{spt}_\omega(n)$ is the total number of smallest parts in all partitions of n in which each odd part is less than twice the smallest part. It satisfies the following congruences.

$$\mathrm{spt}_\omega(5n+3) \equiv 0 \pmod 5,$$
$$\mathrm{spt}_\omega(10n+7) \equiv 0 \pmod 5,$$
$$\mathrm{spt}_\omega(10n+9) \equiv 0 \pmod 5.$$

Mock theta functions in the work of Andrews-Dixit-Yee

The third-order mock theta functions discussed in the paper include

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}, \qquad \nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}.$$

$$q\omega(q) = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}$$

Recall the other third-order mock theta function

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}.$$

Recent work of Andrews and Kumar uses another analogue to connect $f(q)$ with positive odd rank and two-color partitions.

Motivation

New spt identities may reveal new partition statistics or congruences.

Their new analogue is obtained using the limiting case of another ${}_{10}\phi_9$-transformation due to Bailey (limiting case due to Andrews).

$$\lim_{N\to\infty} {}_{10}\phi_9\left(\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & r_1, & -r_1, & r_2, & -r_2, & q^{-N}, & -q^{-N} \\ \sqrt{a}, & -\sqrt{a}, & \frac{aq}{b}, & \frac{aq}{r_1}, & -\frac{aq}{r_1}, & \frac{aq}{r_2}, & -\frac{aq}{r_2}, & aq^{N+1}, & -aq^{N+1}; q, & -\frac{a^3q^{3+2N}}{br_1^2r_2^2} \end{matrix}\right)$$

$$= \frac{(a^2q^2;q^2)_\infty \left(\frac{a^2q^2}{r_1^2r_2^2};q^2\right)_\infty}{\left(\frac{a^2q^2}{r_1^2};q^2\right)_\infty \left(\frac{a^2q^2}{r_2^2};q^2\right)_\infty} \sum_{n=0}^{\infty} \frac{(r_1^2;q^2)_n (r_2^2;q^2)_n (-aq/b;q)_{2n}}{(q^2;q^2)_n (a^2q^2/b^2;q^2)_n (-aq;q)_{2n}} \left(\frac{a^2q^2}{r_1^2r_2^2}\right)^n.$$

Using this, Andrews and Kumar proved

$$\sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2(-q^{n+1};q)_n(q^{n+1};q)_\infty}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} + \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)}{(1-q^{2n})^2} q^{3n(n+1)/2}.$$

Analogue from Andrews-Kumar and implications (PAMS, 2025)

Theorem

Let $N_e(n)$ denote the number of partitions of n with even rank. Then,

$$\sum_{n=1}^{\infty} N_e(n)q^n = \frac{1}{(q;q)_\infty} - 2\sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})(-q^{n+1};q)_n(q^{n+1};q)_\infty}$$

$$f_3(q) = \frac{1}{(q;q)_\infty} - 4\sum_{n=1}^{\infty} \frac{q^{2n}}{(q^{2n};q^2)_n(-q^{2n+1};q)_\infty}$$

The partition theoretic interpretation of this is the following. Let $N_o^+(n)$ denote the number of partitions of n with positive odd rank (the rank of a partition is the largest part minus the number of parts), and let $G(n)$ denote the number of partitions into red and blue parts, where the smallest part is even (say $2m$) and all red parts are even and lie within the interval $[2m, 4m]$.

Theorem

$$N_o^+(n) = G(n)$$

The new source - Verma-Jain transformations (JMAA, 1980)

Pattern

The known analogues had been produced using ${}_{10}\phi_9$-transformations. We move on to higher-parameter transformations.

We derive new analogues using two ${}_{12}\phi_{11}$-transformations of Verma and Jain.

- The first transformation uses parameters grouped as

$$x, \omega x, \omega^2 x, \qquad y, \omega y, \omega^2 y,$$

  where ω is a primitive cube root of unity.
- The second transformation uses shifted triples

$$x, xq, xq^2, \qquad y, yq, yq^2.$$

Why triples matter

Products over the three cube roots of unity factors collapse to base q^3 products, which is why the final identities contain $(\cdot; q^3)_n$ and $(\cdot; q^3)_\infty$.

First Verma-Jain transformation ⇒ First spt-function identity analogue

With $\omega^3 = 1$, the first identity we use is

$$
{}_{12}\phi_{11}\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, x, \omega x, \omega^2 x, y, \omega y, \omega^2 y, q^{-N}, \omega q^{-N}, \omega^2 q^{-N} \\ \sqrt{a}, -\sqrt{a}, aq/x, aq/\omega x, aq/\omega^2 x, aq/y, aq/\omega y, aq/\omega^2 y, aq^{N+1}, \omega^2 aq^{N+1}, \omega aq^{N+1} \end{matrix}; q, \frac{a^4 q^{4+3N}}{x^3 y^3}\right)
$$

$$
= \frac{(a^3q^3;q^3)_N (a^3q^3/x^3y^3;q^3)_N}{(a^3q^3/x^3;q^3)_N (a^3q^3/y^3;q^3)_N} {}_6\phi_5\left(\begin{matrix} q^{-3N}, x^3, y^3, aq, aq^2, aq^3 \\ (aq)^{3/2}, -(aq)^{3/2}, a^{3/2}q^3, -a^{3/2}q^3, x^3y^3q^{-3N}/a^3 \end{matrix}; q^3, q^3\right).
$$

The identity (S., Bull. Austral. Math. Soc., 2026)

$$
\sum_{n=1}^{\infty} \frac{q^{3n}}{(1-q^{3n})^2 (q^{3(n+1)};q^3)_n (q^{3n+1};q)_\infty} = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{nq^{3n}}{1-q^{3n}} + \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n)}{(1-q^{3n})^2} q^{\frac{n(3n+5)}{2}}
$$

- The left side has an spt-type quadratic factor $(1-q^{3n})^{-2}$.
- The first right-hand series is a Lambert series with $q \mapsto q^3$.

The identity

$$\sum_{n=1}^{\infty} \frac{q^{3n}}{(1-q^{3n})^2(q^{3(n+1)};q^3)_n(q^{3n+1};q)_\infty} = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}} + \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n)}{(1-q^{3n})^2}q^{\frac{n(3n+5)}{2}}$$

- The summand
$$\frac{q^{3n}}{(1-q^{3n})^2(q^{3(n+1)};q^3)_n(q^{3n+1};q)_\infty}$$
has a direct resemblance to Andrews' original summand
$$\frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty}.$$
and has a quadratic factor $(1-q^{3n})^{-2}$

- The first right-hand series is a Lambert series with $q \mapsto q^3$.

Second Verma-Jain transformation $\Rightarrow$ Second spt-function identity analogue

The second Verma-Jain identity we use is

$$
{}_{12}\phi_{11}\left(\begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, x, xq, xq^2, y, yq, yq^2, q^{-N}, q^{-N+1}, q^{-N+2} \\ \sqrt{a}, -\sqrt{a}, aq^3/x, aq^2/x, aq/x, aq^3/y, aq^2/y, aq/y, aq^{N+3}, aq^{N+2}, aq^{N+1} \end{matrix}; q^3, \frac{a^4q^{3+3N}}{x^3y^3}\right)
$$

$$
= \frac{(aq;q)_N(aq/xy;q)_N}{(aq/x;q)_N(aq/y;q)_N}\,{}_6\phi_5\left(\begin{matrix} \sqrt[3]{a}, \omega\sqrt[3]{a}, \omega^2\sqrt[3]{a}, x, y, q^{-n} \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, xyq^{-n}/a \end{matrix}; q, q\right).
$$

Here the transformation goes in the opposite base direction - the ${}_{12}\phi_{11}$-side is on base q^3, while the ${}_6\phi_5$ side is on base q.

The identity (S., Bull. Austral. Math. Soc., 2026)

$$
\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)(q^{3n};q^3)_\infty(-q;q)_{n-1}(q;q^2)_n} = \frac{1}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \frac{1}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^{3n})}{(1-q^{3n})^2}q^{3n(3n+1)/2}
$$

Proof template

Both proofs follow the same template -

1. Start from the appropriate Verma-Jain ${}_{12}\phi_{11}$-transformation.
2. Let $a \to 1$, and set $x = z$, $y = z^{-1}$.
3. Use cube-root or shifted-triple products to collapse terms into base q^3 products.
4. Let $N \to \infty$, using an explicit finite-product limit.
5. Rearrange the result into a z-dependent identity.
6. Apply $-\frac{1}{2}\frac{d^2}{dz^2}$ at $z = 1$.
7. Divide by $(q;q)_\infty$ or $(q^3;q^3)_\infty$ and rewrite the left side.

The derivative identity that creates spt weights

The key calculus identity, due to Andrews, used in both proofs, is

$$-\frac{1}{2}\frac{d^2}{dz^2}\left((1-z)(1-z^{-1})f(z)\right)\Big|_{z=1} = f(1).$$

After the substitutions $x = z$, $y = z^{-1}$, the hypergeometric identities contain products like

$$(z;q^r)_n(z^{-1};q^r)_n \quad \text{or} \quad (z;q^r)_\infty(z^{-1};q^r)_\infty.$$

These products have a double zero at $z = 1$. The second derivative extracts the contribution that becomes the square denominator $(1-q^{rn})^{-2}$.

Takeaway

This is the mechanism that turns a transformation of basic hypergeometric series into an spt-function identity.

Proof of first identity

$$\sum_{n=1}^{\infty}\frac{q^{3n}}{(1-q^{3n})^2(q^{3(n+1)};q^3)_n(q^{3n+1};q)_\infty}=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}+\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n)}{(1-q^{3n})^2}q^{\frac{n(3n+5)}{2}}$$

For $n \geq 1$, the proof uses

$$\lim_{a\to 1}\frac{(a;q)_n(q\sqrt{a};q)_n(-q\sqrt{a};q)_n}{(\sqrt{a};q)_n(-\sqrt{a};q)_n}=(q;q)_n(1+q^n).$$

Then in the first Verma-Jain transformation

$${}_{12}\phi_{11}\left(\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, x, \omega x, \omega^2 x, y, \omega y, \omega^2 y, q^{-N}, \omega q^{-N}, \omega^2 q^{-N}\\ \sqrt{a}, -\sqrt{a}, aq/x, aq/\omega x, aq/\omega^2 x, aq/y, aq/\omega y, aq/\omega^2 y, aq^{N+1}, \omega^2 aq^{N+1}, \omega aq^{N+1}\end{matrix};q,\frac{a^4q^{4+3N}}{x^3y^3}\right)$$

$$=\frac{(a^3q^3;q^3)_N(a^3q^3/x^3y^3;q^3)_N}{(a^3q^3/x^3;q^3)_N(a^3q^3/y^3;q^3)_N}\,{}_6\phi_5\left(\begin{matrix}q^{-3N}, x^3, y^3, aq, aq^2, aq^3\\ (aq)^{3/2}, -(aq)^{3/2}, a^{3/2}q^3, -a^{3/2}q^3, x^3y^3q^{-3N}/a^3\end{matrix};q^3,q^3\right),$$

one sets

$$a\to 1,\qquad x=z,\qquad y=z^{-1}.$$

This produces factors

$$(z)_n(\omega z)_n(\omega^2 z)_n(z^{-1})_n(\omega z^{-1})_n(\omega^2 z^{-1})_n,$$

which later collapse to base q^3 factors.

Proof of first identity - root-of-unity collapse, the $N \to \infty$ limit and the key z-identity

The cube-root grouping is essential. Since $\omega^3 = 1$, products over

$$z, \quad \omega z, \quad \omega^2 z$$

collapse into base q^3 expressions. In the proof this is why the final rearranged identity contains

$$(z^3; q^3)_n (z^{-3}; q^3)_n.$$

After replacing z^3 by z, we arrive at expressions in $(z; q^3)_n (z^{-1}; q^3)_n$. The finite truncation terms satisfy

$$\lim_{N\to\infty} \frac{(q^{-N})_n (\omega q^{-N})_n (\omega^2 q^{-N})_n}{(q^{N+1})_n (\omega^{-1} q^{N+1})_n (\omega^{-2} q^{N+1})_n} q^{3Nn} = (-1)^n q^{3n(n-1)/2}.$$

After the specialization, limit, simplification, and replacement $z^3 \mapsto z$, we obtain

$$\sum_{n=0}^{\infty} \frac{(z; q^3)_n (z^{-1}; q^3)_n (q; q)_{3n}}{(q^3; q^3)_{2n} (q^3; q^3)_n} q^{3n}$$

$$= \frac{1}{(q^3; q^3)_\infty^2} \left((zq^3; q^3)_\infty (z^{-1} q^3; q^3)_\infty + \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n)(z; q^3)_\infty (z^{-1}; q^3)_\infty}{(1 - zq^{3n})(1 - z^{-1} q^{3n})} q^{n(3n+5)/2} \right)$$

Proof of the first identity - derivative evaluations

Key z-identity

$$\sum_{n=0}^{\infty} \frac{(z;q^3)_n (z^{-1};q^3)_n (q;q)_{3n}}{(q^3;q^3)_{2n}(q^3;q^3)_n} q^{3n}$$

$$= \frac{1}{(q^3;q^3)_\infty^2} \left((zq^3;q^3)_\infty (z^{-1}q^3;q^3)_\infty + \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n)(z;q^3)_\infty (z^{-1};q^3)_\infty}{(1-zq^{3n})(1-z^{-1}q^{3n})} q^{n(3n+5)/2} \right)$$

We differentiate the above identity twice using

$$-\frac{1}{2}\frac{d^2}{dz^2}\left((z;q^3)_n (z^{-1};q^3)_n\right)\Big|_{z=1} = (q^3;q^3)_{n-1}^2,$$

$$-\frac{1}{2}\frac{d^2}{dz^2}\left(\frac{(z;q^3)_\infty (z^{-1};q^3)_\infty}{(1-zq^{3n})(1-z^{-1}q^{3n})}\right)\Big|_{z=1} = \frac{(q^3;q^3)_\infty^2}{(1-q^{3n})^2},$$

$$-\frac{1}{2}\frac{d^2}{dz^2}\left((zq^3;q^3)_\infty (z^{-1}q^3;q^3)_\infty\right)\Big|_{z=1} = (q^3;q^3)_\infty^2 \sum_{n=1}^{\infty} \frac{nq^{3n}}{1-q^{3n}}.$$

All follow from

$$-\frac{1}{2}\frac{d^2}{dz^2}\left((1-z)(1-z^{-1})f(z)\right)\Big|_{z=1} = f(1).$$

Proof of the first identity

The differentiated identity is

$$\sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}^2 (q;q)_{3n}}{(q^3;q^3)_{2n}(q^3;q^3)_n} q^{3n} = \sum_{n=1}^{\infty} \frac{nq^{3n}}{1-q^{3n}} + \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)}{(1-q^{3n})^2} q^{n(3n+5)/2}.$$

The left side after division by $(q;q)_\infty$ is

$$\frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}^2 (q;q)_{3n}}{(q^3;q^3)_{2n}(q^3;q^3)_n} q^{3n}.$$

Product cancellation rewrites it as

$$\sum_{n=1}^{\infty} \frac{q^{3n}}{(1-q^{3n})^2 (q^{3(n+1)};q^3)_n (q^{3n+1};q)_\infty}.$$

$$\sum_{n=1}^{\infty} \frac{q^{3n}}{(1-q^{3n})^2 (q^{3(n+1)};q^3)_n (q^{3n+1};q)_\infty} = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{nq^{3n}}{1-q^{3n}} + \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)}{(1-q^{3n})^2} q^{\frac{n(3n+5)}{2}}$$

Proof of the second identity

For the second identity, the proof uses two limits.

$$\lim_{a\to 1}\frac{(a;q^3)_n(q^3\sqrt{a};q^3)_n(-q^3\sqrt{a};q^3)_n}{(\sqrt{a};q^3)_n(-\sqrt{a};q^3)_n}=(q^3;q^3)_n(1+q^{3n}).$$

$$\lim_{a\to 1}\frac{(\sqrt[3]{a};q)_n(\omega\sqrt[3]{a};q)_n(\omega^2\sqrt[3]{a};q)_n}{(\sqrt{a};q)_n(-\sqrt{a};q)_n}=\frac{(q^3;q^3)_{n-1}}{(q^2;q^2)_{n-1}}.$$

The finite truncation terms satisfy

$$\lim_{N\to\infty}\frac{(q^{-N};q^3)_n(q^{-N+1};q^3)_n(q^{-N+2};q^3)_n}{(q^{N+3};q^3)_n(q^{N+2};q^3)_n(q^{N+1};q^3)_n}q^{3Nn}=(-1)^n q^{3n(3n-1)/2}.$$

After specialization and simplification, we obtain

$$1+\sum_{n=1}^{\infty}\frac{(z;q)_n(z^{-1};q)_n(q^3;q^3)_{n-1}}{(q;q)_n(q^2;q^2)_{n-1}(q;q^2)_n}q^n$$

$$=\frac{1}{(q;q)_\infty^2}\left((zq;q)_\infty(z^{-1}q;q)_\infty+\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^{3n})(z;q)_\infty(z^{-1};q)_\infty}{(1-zq^{3n})(1-z^{-1}q^{3n})}q^{3n(3n+1)/2}\right).$$

The three derivatives used are

$$-\frac{1}{2}\frac{d^2}{dz^2}\big((z;q)_n(z^{-1};q)_n\big)\Big|_{z=1} = (q;q)_{n-1}^2,$$

$$-\frac{1}{2}\frac{d^2}{dz^2}\left(\frac{(z;q)_\infty(z^{-1};q)_\infty}{(1-zq^{3n})(1-z^{-1}q^{3n})}\right)\Big|_{z=1} = \frac{(q;q)_\infty^2}{(1-q^{3n})^2},$$

$$-\frac{1}{2}\frac{d^2}{dz^2}\big((zq;q)_\infty(z^{-1}q;q)_\infty\big)\Big|_{z=1} = (q;q)_\infty^2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

Differentiating twice gives

$$\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}^2(q^3;q^3)_{n-1}}{(q;q)_n(q^2;q^2)_{n-1}(q;q^2)_n}q^n = \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \sum_{n=1}^{\infty}\frac{(-1)^n(1+q^{3n})}{(1-q^{3n})^2}q^{3n(3n+1)/2}.$$

Dividing by $(q^3;q^3)_\infty$ and simplifying products gives the identity.

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)(q^{3n};q^3)_\infty(-q;q)_{n-1}(q;q^2)_n} = \frac{1}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \frac{1}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^{3n})}{(1-q^{3n})^2}q^{3n(3n+1)/2}$$

The two ${}_{12}\phi_{11}$ analogues - a comparison with the ${}_{10}\phi_9$ analogues

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}=\frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+\frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^{2n})}{(1-q^{2n})^2}q^{n(3n+1)}$$

$$\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2(-q^{n+1};q)_n(q^{n+1};q)_\infty}=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}+\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n)}{(1-q^{2n})^2}q^{3n(n+1)/2}$$

There is a visible symmetry between the new identities below and the identities above.

$$\sum_{n=1}^{\infty}\frac{q^{3n}}{(1-q^{3n})^2(q^{3(n+1)};q^3)_n(q^{3n+1};q)_\infty}=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}+\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n)}{(1-q^{3n})^2}q^{\frac{n(3n+5)}{2}}$$

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)(q^{3n};q^3)_\infty(-q;q)_{n-1}(q;q^2)_n}=\frac{1}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+\frac{1}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^{3n})}{(1-q^{3n})^2}q^{3n(3n+1)/2}$$

The second sum on the RHS of the identity above resembles the second sum in the first analogue after a formal change of scale.

Concluding observations - possible mock theta or other links

In earlier works, nonweighted companions of spt identities led to mock theta functions.

$$q\omega(q) = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}$$

$$f_3(q) = \frac{1}{(q;q)_\infty} - 4\sum_{n=1}^{\infty} \frac{q^{2n}}{(q^{2n};q^2)_n(-q^{2n+1};q)_\infty}$$

Proofs make use of the following four-parameter deep identity due to Andrews.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(B;q)_n(-Abq;q)_n q^n}{(-aq;q)_n(-bq;q)_n} \\
&= \frac{-a^{-1}(B;q)_\infty(-Abq;q)_\infty}{(-bq;q)_\infty(-aq;q)_\infty} \sum_{m=0}^{\infty} \frac{(A^{-1};q)_m \left(\frac{Abq}{a}\right)^m}{\left(-\frac{B}{a};q\right)_{m+1}} \\
&\quad + (1+b)\sum_{m=0}^{\infty} \frac{(-a^{-1};q)_{m+1}\left(-\frac{ABq}{a};q\right)_m (-b)^m}{\left(-\frac{B}{a};q\right)_{m+1}\left(\frac{Abq}{a};q\right)_{m+1}}.
\end{aligned}$$

Concluding observations - A nonweighted companion to first identity

Do the new identities have similar mock theta or other partition-theoretic companions?

Main difficulty

The natural nonweighted versions now have more parameters, so Andrews' four-parameter identity does not immediately simplify them.

Removing one factor $(1-q^{3n})^{-1}$ from the LHS leads to the simplification

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^{3n}}{(1-q^{3n})(q^{3(n+1)};q^3)_n(q^{3n+1};q)_\infty} &= \sum_{n=0}^{\infty}\frac{q^{3n+3}}{(q^{3n+3};q^3)_{n+2}(q^{3n+4};q)_\infty}\\ &= \frac{q^3}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(q;q)_{3n+3}}{(q^{3n+3};q^3)_{n+2}}q^{3n}\\ &= \frac{q^3}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(q^3;q^3)_n(q;q^3)_{n+1}(q^2;q^3)_{n+1}}{(q^3;q^6)_{n+1}(-q^3;q^3)_{n+1}}q^{3n}.\end{aligned}$$

Higher-parameter identities analogous to Andrews' four-parameter identity may be needed to deal with this q-series. See G. E. Andrews et al., Overpartitions related to the mock theta function $\omega(q)$, Acta Arith. **181** (2017), no. 3, 253–286.

Concluding observations - more analogues to discover

There are several possible sources for further identities -

- additional Verma-Jain identities;
- a ${}_{14}\phi_{13}$-identity and a related higher-parameter identity;
- identities due to McLaughlin and Zimmer arising from the WP-Bailey tree;
- multisum generalizations of Verma-Jain's ${}_{14}\phi_{13}$-identities.

Summary of the mathematical narrative

1. Andrews' spt(n) connects smallest-part weights with rank moments and modular phenomena.
2. Earlier analogues came from ${}_{10}\phi_9$-transformations and led to mock theta and two-colour partition interpretations.
3. We obtain two new analogues from Verma-Jain ${}_{12}\phi_{11}$-transformations.
4. The observations point toward possible mock theta connections, higher-parameter identities, and refined partition interpretations.

Possible discussion questions

- Can the two new left-hand sides be given natural partition-theoretic interpretations?
- Do the nonweighted companion series connect to known or new mock theta functions?
- Is there a general framework explaining the symmetry among Andrews-type spt identities?

Thank you!

References

- G. E. Andrews, Ramanujan's "lost" notebook IV. Stacks and alternating parity in partitions, *Adv. Math.* 53 (1984), 55–74.
- G. E. Andrews, The number of smallest parts in the partitions of n, *J. Reine Angew. Math.* 624 (2008), 133–142.
- G. E. Andrews, A. Dixit and A. J. Yee, *Partitions associated with the Ramanujan/Watson mock theta function* $\omega(q)$, $\nu(q)$ *and* $\phi(q)$, Res. Number Theory **1** (2015), 1–25.
- G. E. Andrews, A. Dixit, D. Schultz and A. J. Yee, Overpartitions related to the mock theta function $\omega(q)$, *Acta Arith.* 181 (2017), 253–286.
- G. E. Andrews and R. Kumar, Rank, two-color partitions and mock theta functions, *Proc. Amer. Math. Soc.* 153 (2025), 4669–4682.
- R. Sarma, Further analogues of Andrews' spt-function identity, *Bull. Aust. Math. Soc.*, published online 2026.
- A. Verma and V. K. Jain, Transformations between basic hypergeometric series on different bases and identities of Rogers-Ramanujan type, *J. Math. Anal. Appl.* 76 (1980), 230–269.